An old problem of Erdős: a graph without two cycles of the same length

Chunhui Lai *

School of Mathematics and Statistics, Minnan Normal University,

Zhangzhou, Fujian 363000, P. R. of CHINA

laich2011@msn.cn; laichunhui@mnnu.edu.cn

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph on n vertices in which any two cycles are of different lengths. Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths. Let M_n be the set of simple graphs on n vertices and $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. In this paper, it is proved that for n sufficiently large, $mc(n) \leq \frac{15}{16}n$.

We make the following conjecture:

Conjecture.

$$\lim_{n\to\infty} \frac{mc(n)}{n} = 0.$$

1 Introduction

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, P. Erdős raised the problem of determining $f(n)$ (see [1], p.247, Problem 11). Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths. Let M_n be the set of simple graphs on n vertices and $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. Let $sc(n)$ be the second-largest cycle length for all $G \in M_n$. Let $tc(n)$ be the third-largest cycle length for all $G \in M_n$. A natural question is what is the numbers of $mc(n)$,

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$sc(n)$, $tc(n)$. Let $mcn(n)$ be the maximum cycle numbers for all $G \in M_n$. A natural question is what is the numbers of $mcn(n)$. Let $b(n)$ be the maximum 2-connected block numbers for all $G \in M_n$. A natural question is what is the numbers of $b(n)$. Shi[23] proved that

Theorem 1 (Shi [23]).

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$ and $f(n) = f^*(n-1)+3$ for $n \geq 3$.

Lai[8] proved that

Theorem 2 (Lai [8]). For $n \geq e^{2m}(2m+3)/4$,

$$f(n) < n - 2 + \sqrt{nln(4n/(2m+3)) + 2n} + log_2(n+6).$$

Chen, Lehel, Jacobson and Shreve[3] gave a quick proof of this result.

Jia[6], Lai[7,8,9,10,11,12,13,14,15], Shi[23,24,25,26,27,28], Shi, Tang, Tang, Gong, Xu[29], Shi, Xu, Chen, Wang[30] obtained some additional related results.

Lai[16] proved that

Theorem 3 (Lai [16]).

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

and Lai[9] conjectured that

conjecture 4 (Lai [9]).

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3}.$$

Boros, Caro, Füredi and Yuster[2] proved that

Theorem 5 (Boros, Caro, Füredi and Yuster[2]).

$$f(n) \leq n + 1.98\sqrt{n}(1+o(1)).$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

In 1988, Shi[23] proved that

Theorem 6 (Shi[23]). For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n-15}-3)]$.

In 1998, G. Chen, J. Lehel, M. S. Jacobson, and W. E. Shreve [3] proved that

Theorem 7 (Chen, Lehel, Jacobson and Shreve [3]). $f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

In 2001, E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] improved this lower bound significantly:

Theorem 8 (Boros, Caro, Füredi and Yuster[2]). $f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$.

and conjectured that

Conjecture 9 (Boros, Caro, Füredi and Yuster[2]). $\lim \frac{f_2(n)-n}{\sqrt{n}} = 1$.

It is easy to see that this Conjecture implies the (difficult) upper bound in the Erdős Turán Theorem [4][5](see [2]).

Markström [22] raised the problem

Problem 10 (Markström [22]). Determining the maximum number of edges in a Hamiltonian graph on n vertices with no repeated cycle lengths.

Let $g(n)$ be the maximum number edges in an n-vertex, Hamiltonian graph with no repeated cycle lengths. J. Lee, C. Timmons [18] proved the following.

Theorem 11 (J. Lee, C. Timmons [18]). If q is a power of a prime and $n = q^2 + q + 1$, then

$$g(n) \geq n + \sqrt{n - 3/4} - 3/2$$

A simple counting argument shows that $g(n) < n + \sqrt{2n} + 1$.

Let MH_n be the set of Hamiltonian graphs on n vertices and $g(n)$ edges in which any two cycles are of different lengths. Let $mcn_H(n)$ be the maximum cycle numbers for all $G \in MH_n$. A natural question is what is the numbers of $mcn_H(n)$.

J. Ma, T. Yang [21] proved that

Theorem 12 (Ma, Yang [21]). Any n-vertex 2-connected graph with no two cycles of the same length contains at most $n + \sqrt{n} + o(\sqrt{n})$ edges.

Let $f_2(n,k)$ be the maximum number of edges in a graph G on n verticesin which no two cycles have the same length and G which consists of k 2-connected blocks. A natural question is what is the maximum number of edges $f_2(n,k)$. It is clearly that $f_2(n,1) = f_2(n)$.

By theorem 5, it is clearly that

$$f_2(n,k) \leq f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

H. Lin, M. Zhai,Y. Zhao [19] proved that

Theorem 13 (Lin, Zhai,Zhao [19]). Let G be a graph of order $n \geq 26$. If $\rho(G) \geq \rho(K^+_{1,n-1})$, then G contains two cycles of the same length unless $G \cong K^+_{1,n-1}$.

and asked the following problem.

Problem 14 (Lin, Zhai,Zhao [19]). What is the maximum spectral radius among all 2-connected n-vertex graphs without two cycles of the same length?

Y. Shi [27]proved that

Theorem 15 (Shi [27]).

$$b(n) \leq [(\sqrt{8n+1} - 5)/2] + 1$$

C. Lai [7] proved that

Theorem 16 (Lai [7]). $mc(n) \leq n-1$ for $n \geq \sum_{i=1}^{71} i - 8 \times 18$.

Survey papers on this problem can be found in Tian[31], Zhang[32], Lai and Liu[17].

The progress of all 50 problems in [1] can be found in Locke[20]. Let $v(G)$ denote the number of vertices, and $\varepsilon(G)$ denote the number of edges. In this paper, it is proved that

Theorem 17. For n sufficiently large,

$$mc(n) \leq \frac{15}{16}n.$$

2 Proof of the theorem 17

Proof. If $mc(n) > \frac{15}{16}n$, for n sufficiently large, then there is a simple graph G on n vertices and $f^*(n)$ edges in which any two cycles are of different lengths, the maximum cycle length of G is $mc(n)$. Let G_1 be the block contain the cycle with length $mc(n)$. It is clear that $v(G_1) > \frac{15}{16}n$. By the result of Ma and Yang [21], $\varepsilon(G_1) \leq v(G_1) + \sqrt{v(G_1)} + o(\sqrt{v(G_1)})$. By the result of Boros, Caro, Füredi and Yuster [2], $\varepsilon(G) \leq v(G_1)+\sqrt{v(G_1)}+o(\sqrt{v(G_1)})+V(G)-V(G_1)+1+1.98\sqrt{V(G)-V(G_1)+1}(1+o(1)) \leq n+1+\sqrt{n}+o(\sqrt{n})+1.98\sqrt{\frac{1}{16}n}(1+o(1)) \leq n+\frac{3}{2}\sqrt{n}$, for n sufficiently large. By the result of Shi [23] and Lai [16], $\varepsilon(G) = f^*(n) = f(n+1)-3 > n+(\sqrt{2+\frac{40}{99}}-o(1))\sqrt{n}$, for n sufficiently large. Note that $\varepsilon(G) \leq n+\frac{3}{2}\sqrt{n}$, this contradiction completes the proof.

It is clear that $mcn(n) \leq mc(n) - 2$.

By theorem 3, it is clearly that

$$mcn(n) \geq \sqrt{2+\frac{40}{99}}\sqrt{n}(1-o(1)).$$

We make the following conjecture:

Conjecture.

$$\lim_{n\to\infty} \frac{mc(n)}{n} = 0.$$

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